A CRITERION FOR THE COVERING CONDITION OF GENERALIZED RANDOM MATRIX ENSEMBLES

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Abstract. In this paper we present a criterion for the covering condition of the generalized random matrix ensemble, which enable us to verify the covering condition for the seven classes of generalized random matrix ensemble in an unified and simpler way.

1. Introduction

In [1, 2] the authors introduced the concept of generalized random matrix ensemble, and presented a classification scheme of it by means of Lie theory, namely linear ensemble, nonlinear noncompact ensemble, compact ensemble, group ensemble, algebra ensembles, pseudo-group ensemble, and pseudo-algebra ensemble. The seven classes of generalized ensemble include all classical random matrix ensembles(see [3, 4, 6]) and some new ensembles, the joint density functions of which were derived in an unified way. Various kinds of classical integration formulae (Weyl integration formula for compact Lie groups, Harish-Chandra's integration formulae for complex semisimiple Lie groups and real reductive groups, integration formulae associated with symmetric spaces of noncompact and compact types, and their linear versions) were also deduced as corollaries of an integration formula associated with the generalized random matrix ensemble. The integration formula associated with generalized ensemble relies on a covering condition for the generalized ensemble, which asserts that a natural map associated with the generalized ensemble is a finite sheeted covering map. Two criterions of the covering condition were proved in [1], and were all used in a confusing way when the authors verified the covering condition for various kinds of generalized ensembles. This drawback somewhat conflicts to the goal of the authors in [1, 2] which was to present an unified theory of random matrix ensembles. The goal of this paper is to fix this drawback. We present a criterion for the covering condition of the generalized random matrix ensemble, which enable us to verify the covering condition for the seven classes of generalized ensemble in an unified way, and the verifications for all of the seven cases are simpler than that of in [2].

Suppose a Lie group G acts on a Riemannian manifold X by $\sigma : G \times X \to X$, and suppose the induced Riemannian measure dx is G-invariant. Let Y be a closed submanifold of X with the induced Riemannian measure dy, and let $K = \{g \in G : \sigma_g(y) = y, \forall y \in Y\}$. Define the map $\varphi : G/K \times Y \to X$ by $\varphi([g], y) = \sigma_g(y)$. Let $X_z \subset X$, $Y_z \subset Y$ be closed subsets of measure zero in X and Y, respectively. Denote $X' = X \setminus X_z$, $Y' = Y \setminus Y_z$. Suppose the following conditions hold.

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(a) (*Invariance condition*) $X' = \bigcup_{y\in Y'} O_y$.
(b) (*Transversality condition*) $T_yX = T_yO_y \oplus T_yY, \quad \forall y \in Y'$.
(c) (*Dimension condition*) $\dim G_y = \dim K, \quad \forall y \in Y'$.
(d) (*Orthogonality condition*) $T_yY \perp T_yO_y, \quad \forall y \in Y'$.
(e) (*Covering condition*) *There is a positive integer d such that the map $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map.*

Suppose $d\mu$ is a G-invariant smooth measure on G/K, and suppose $p(x)$ is a G-invariant smooth function on X. Then the system $(G, \sigma, X, p(x)dx, Y, dy)$ is a generalized random matrix ensemble. It is proved in [1] that there is a generalized eigenvalue distribution $d\nu$ on Y such that $\varphi^*(p(x)dx) = d\mu d\nu$, and we have the integration formula

$$\int_X f(x)p(x)dx = \frac{1}{d}\int_Y \left(\int_{G/K} f(\sigma_g(y))d\mu([g])\right) d\nu(y) \tag{1.1}$$

for all $f \in C^\infty(X)$ with $f \geq 0$ or with $f \in L^1(X, p(x)dx)$.

Let $N = \{g \in G : \sigma_g(Y) = Y\}$. Then N is a closed subgroup of G, K is a normal subgroup of N. For $y \in Y$, denote $N_y = \{g \in G : \sigma_g(y) \in Y\}$. The main result in this paper is the following conclusion.

Theorem 1.1. *Suppose conditions (a), (b) and (c) hold. If N/K is a finite group of order d, and $N_y = N$ for every $y \in Y'$, then the covering condition (e) holds, that is, $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map.*

Theorem 1.1 will be proved in Section 2. Using Theorem 1.1, the covering condition for the seven classes of generalized ensemble will be re-verified in Section 3.

2. Proof of Theorem 1.1

We first recall some facts which were proved in [1].

Fact 2.1. ([1], Proposition 3.2.) *Suppose conditions (a), (b) and (c) hold. Then $\varphi : G/K \times Y' \to X'$ is everywhere regular.*

Fact 2.2. ([1], Proposition 3.5.) *Let M, N be smooth manifolds of the same dimension, d a positive integer. Then an everywhere regular smooth map $f : M \to N$ is a d-sheeted covering map if and only if for every $p \in N$, $\varphi^{-1}(p)$ has d points.*

Proof of Theorem 1.1. It can be easily verified from the dimension condition (c) that $\dim(G/K \times Y') = \dim X'$ (see also [1]). By Facts 2.1 and 2.2, it is sufficient to prove that for every $x \in X'$, $\varphi^{-1}(x)$ has d points.

Let $x \in X'$. By the invariance condition (a), there is some $g \in G$ such that $\sigma_g(x) \in Y'$. Denote $y = \sigma_g(x)$. Since $\varphi^{-1}(y) = (l_g \times id)(\varphi^{-1}(x))$, where l_g is the left multiplication of g on G/K, it is sufficient to show $\varphi^{-1}(y)$ has d points.

Now choose $g_1, \cdots, g_d \in N$, one in each coset space of K in N. Then we have $\{([g_i], \sigma_{g_i^{-1}}(y)) : i = 1, \cdots, d\} \subset \varphi^{-1}(y)$. On the other hand, if $([g'], y') \in \varphi^{-1}(y)$, that is, $\sigma_{g'}(y') = y$, then $g' \in N_{y'} = N$. So there is some $i_0 \in \{1, \cdots, d\}$ such that $[g'] = [g_{i_0}]$, and then $([g'], y') = ([g_{i_0}], \sigma_{g_{i_0}^{-1}}(y))$. This shows that $\varphi^{-1}(y) \subset \{([g_i], \sigma_{g_i^{-1}}(y)) : i = 1, \cdots, d\}$. Hence $\varphi^{-1}(y) = \{([g_i], \sigma_{g_i^{-1}}(y)) : i = 1, \cdots, d\}$, which has d points. $\square$

3. Verification of the Covering Condition

The covering condition was verified in [2] for the seven classes of general ensembles respectively. One ingredient was Corollary 3.6 in [1], which said that if $|O_y \cap Y'| = d$ and the isotropic $G_y = K$ for every $y \in Y'$, then the covering condition holds. But this criterion was not always available. For example, when verifying the covering condition for the compact ensemble and the group ensemble associated with complex semisimple Lie groups, the authors had to appeal to a Proposition 3.5 in [1], that is Fact 2.2 in this paper. In this section we reverify the covering conditions for the seven classes of generalized ensembles using Theorem 1.1. We will see that the process of verification here is simpler than that of in [2] for each of the seven cases.

Theorem 3.1. *The linear ensemble, the nonlinear noncompact ensemble, the compact ensemble, group ensembles associated with compact Lie groups and complex semisimple Lie groups, algebra ensembles associated with compact Lie groups and complex semisimple Lie groups, the pseudo-group ensemble, and the pseudo-algebra ensemble satisfy the covering condition.*

Proof. We adopt the notations in corresponding sections of [2].

(i) Linear ensemble. Since by Proposition 7.32 in [5] the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is finite, by Theorem 1.1, it is sufficient to prove that for $\eta \in \mathfrak{a}'$, if $k \in K$ such that $\eta' = \mathrm{Ad}(k)(\eta) \in \mathfrak{a}'$, then $k \in N_K(\mathfrak{a})$. Since $\mathrm{Ad}(k)$ is an automorphism of $\mathfrak{g}$, $\mathrm{Ad}(k)(Z_{\mathfrak{g}}(\eta)) = Z_{\mathfrak{g}}(\eta')$. But $\eta, \eta' \in \mathfrak{a}' = \mathfrak{a} \setminus (\bigcup_{\lambda \in \Sigma^+} \ker \lambda)$ implies $Z_{\mathfrak{g}}(\eta) = Z_{\mathfrak{g}}(\eta') = \mathfrak{g}_0$. So $\mathrm{Ad}(k)$ fixes $\mathfrak{g}_0$. But $\mathfrak{p}$ is also fixed by $\mathrm{Ad}(k)$. This implies $\mathrm{Ad}(k)$ fixes $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$, that is, $k \in N_K(\mathfrak{a})$, finishing the proof of this case.

(ii) Nonlinear noncompact ensemble. Since $\exp|_{\mathfrak{p}} : \mathfrak{p} \to P$ is a diffeomorphism and $\exp(\mathfrak{a}') = A'$ by definition, this case is equivalent to the case of linear ensemble.

(iii) Compact ensemble. Since $A' = A \setminus (\bigcup_{\lambda \in \Sigma^+} \ker \vartheta_\lambda)$, for $a \in A'$, we have $Z_{\mathfrak{g}}(a) = \mathfrak{g}_0$. Similar to she proof for linear ensemble, if $\sigma_k(a) \in A'$ for some $k \in K, a \in A'$, then $\mathrm{Ad}(k)$ fixes $\mathfrak{a}$, that is, $k \in N_K(\mathfrak{a}) = N_K(A)$.

(iv) Group ensemble and algebra ensemble associated with compact Lie groups. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, T a maximal torus with Lie algebra $\mathfrak{t}$. As in [2], let $T' = T \setminus (\bigcup_{\alpha \in \Delta} \ker \vartheta_\alpha)$, $\mathfrak{t}' = \mathfrak{t} \setminus (\bigcup_{\alpha \in \Delta} \ker \alpha)$. Then for $t \in T'$ and $\eta \in \mathfrak{t}'$, $Z_{\mathfrak{g}}(t) = Z_{\mathfrak{g}}(\eta) = \mathfrak{t}$. So if $t \in T'$ and $g \in G$ such that $gtg^{-1} \in T'$, then $\mathrm{Ad}(g)(\mathfrak{t}) = \mathfrak{t}$, that is, $g \in N_G(\mathfrak{t})$. Similarly, if $\eta \in \mathfrak{t}'$ and $g \in G$ such that $\mathrm{Ad}(g)(\eta) \in \mathfrak{t}'$, then $g \in N_G(\mathfrak{t})$. By Theorem 1.1, the associated ensembles satisfy the covering condition with covering sheet $|W|$.

(v) Group ensemble and algebra ensemble associated with complex semisimple Lie groups. It is almost same as the proof of case (iv).

(vi) Pseudo-group ensemble and pseudo-algebra ensemble. Let G be a real reductive group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}_j$ a Cartan subalgebra of $\mathfrak{g}$, $H_j = Z_G(\mathfrak{h}_j)$ the associated Cartan subgroup of G. As in [2], let $\mathfrak{h}_j' = \mathfrak{h}_j \cap \mathfrak{g}_r$, $H_j' = H_j \cap G_r$. For the associated pseudo-algebra ensemble, if $\eta \in \mathfrak{h}_j'$ and $g \in G$ such that $\mathrm{Ad}(g)(\eta) \in \mathfrak{h}_j'$, then $\mathrm{Ad}(g)(\mathfrak{h}_j) = \mathfrak{h}_j$, due to the fact that $Z_{\mathfrak{g}}(\eta) = \mathfrak{h}_j$. By Theorem 1.1, the covering condition is satisfied with covering sheet $|W_j| = |N_G(\mathfrak{h}_j)/H_j|$. For the associated pseudo-algebra ensemble, if $h \in H'$ and $g \in G$ such that $ghg^{-1} \in H'$, then $\mathrm{Ad}(g)(\mathfrak{h}_j) = \mathfrak{h}_j$, due to the fact that $Z_{\mathfrak{g}}(h) = \mathfrak{h}_j$. Hence $g \in N_G(H_j)$, and the covering condition is satisfied with covering sheet $|N_G(H_j)/Z_G(H_j)| = |W_j| \cdot |H_j/Z(H_j)|$, which is finite by Proposition 7.25 in [5]. $\square$

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